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COMPOSITE RIBBON NUMBER ONE KNOTS HAVE TWO-BRIDGE SUMMANDS

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ABSTRACT. A composite ribbon knot which can be sliced with a single band move has a two-bridge summand.

A knot K in the 3-sphere is said to be of *ribbon number* r if K is a ribbon knot which cannot be "sliced" in fewer than r band moves; i.e. r is the minimal number of saddle points on a ribbon disc in the 4-ball having as boundary the knot K. A natural question to ask is if a ribbon number one knot is prime. The answer, of course, is no, as the square knot shows. However, the answer is "nearly" yes. Recall that if a knot K is prime then in every decomposition of K into a connected sum the structure of one of the summands is trivial, i.e. there must be an unknotted summand, or, put another way, a summand with bridge number one. We show for a composite ribbon number one knot, the structure of one of the summands, while possibly nontrivial, must still be "elementary". As advertised in the title, we show

Theorem. *A composite ribbon number one knot has a two-bridge summand.*

The theorem above is a direct corollary of Theorem 1.3 below regarding band sums. The proof of 1.3 uses the combinatorial techniques introduced by M. Scharlemann in [Sc_1] and [Sc_2] to minimize the intersection of planar surfaces in a tangle exterior. A knowledge of these techniques is essential to the understanding of this paper.

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1. THE MAIN THEOREM AND PRELIMINARY ARGUMENTS

Divide a 3-ball into four quadrants by two discs, D_v and D_h, one vertical and the other horizontal. Label the quadrants by the points of the compass NE, NW, SW, SE. Let N be the manifold obtained by attaching two 1-handles to the 3-ball, one connecting NE to SE, the other connecting NW to SW. The

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1-handles are to be thought of as being attached at points on the boundary of a disc $D_\perp$ in the 3-ball, perpendicular to both D_v and D_h. The disc $D_\perp$ divides the boundary of the 3-ball into two-hemispheres, call that part of the two hemispheres lying in ∂N the *front* face and *back* face of ∂N respectively. Similarly, the disc D_h divides the boundary of the 3-ball into two hemispheres, call that part of these two hemispheres lying in ∂N the *top* face and *bottom* face of ∂N respectively.

1.1. Let A_m be an imbedded family of simple closed curves in ∂N consisting of circles $a_1, \cdots, a_m$ (labelled west to east) parallel to ∂D_v. Let B_n be an imbedded family of simple closed curves in ∂N consisting of circles $b_1, \ldots, b_n$ (labelled south to north) parallel to ∂D_h, together with two circles b_+ and b_- which are meridians of the 1-handles (possibly of the same 1-handle); see Figure 1.

1.2. Imbed into disjoint 3-balls in S^3 two copies γ_0, γ_1 of the unknot, and let $b: I \times I \to S^3$ be an imbedding such that $b^{-1}(\gamma_i) = I \times \{i\}$, $i = 1, 0$. The band sum $\gamma_0 \#_b \gamma_1$ of γ_0 and γ_1 is obtained by joining $\gamma_0 - b(I \times \{0\})$ to $\gamma_1 - b(1 \times \{1\})$ by the arcs $b(\partial I \times I)$.

1.3 **Theorem.** *Suppose* γ_0 *and* γ_1 *are unknots in the* 3*-sphere and a certain band sum* $K = \gamma_0 \#_b \gamma_1$ *is composite. Then* K *has a two-bridge summand.*

1.4. Let K be as in 1.3. As in [Sc_1] we obtain an imbedding of a genus two handlebody N into S^3 and planar surfaces $P(Q)$ in closure($S^3 - N$) with boundary A_m (B_n). The surface P arises from a 2-sphere S_p separating γ_0

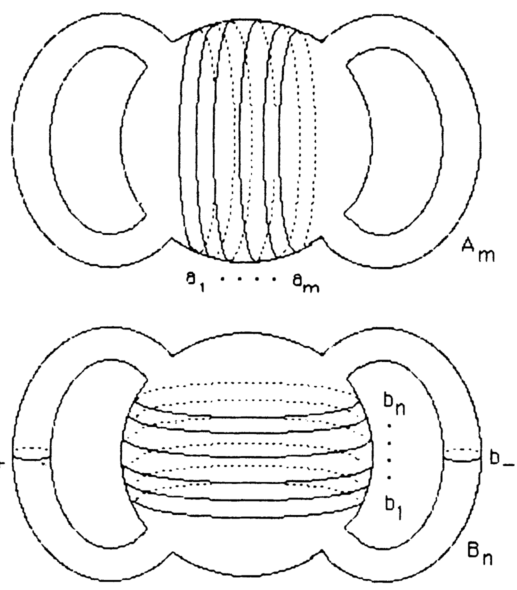


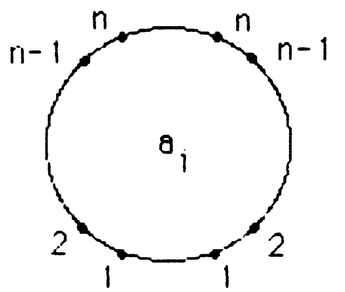
FIGURE 1

from γ_1, the surface Q from a 2-sphere S_Q decomposing K into a nontrivial connected sum. That is, S_Q meets K in two points and separates S^3 into 3-balls B_1 and B_2; each of which meets K in a nontrivial knotted arc. As both γ_0 and γ_1 are unknotted, $n > 0$.

Assume that $m+n$ is minimal with respect to the conditions on the surfaces S_p and S_Q, and that S_Q decomposes K into summands K_1 and K_2, each with bridge number at least three. Further assume that P and Q are in general position so that $P \cap Q$ consists of arcs and circles, and that the number of components of $P \cap Q$ is minimal. In particular, any circle left in $P \cap Q$ is essential in both P and Q.

Construct as follows semioriented graphs Γ_P and Γ_Q in the 2-spheres S_P and S_Q. Regard each a_i (b_s) as a fat vertex in Γ_P (Γ_Q). Regard b_+ and b_- as valence zero vertices in Γ_Q. Finally, regard the arc components of $P \cap Q$ as edges of the graphs Γ_P and Γ_Q. For $1 \leq i \leq m$, $1 \leq s \leq n$ there are two points of intersection of a_i with b_s, each of which label (i, s). The end of an edge in Γ_P (Γ_Q) is assigned the second (first) coordinate of the label above for the corresponding point in $A_m \cap B_n$. In $S_P(S_Q - \{b_+, b_-\})$ therer are $m(n)$ vertices of $\Gamma_P(\Gamma_Q - \{b_+, b_-\})$, each of valence $2n(2m)$. Figure 2 shows how the labels sit around the vertices.

1.5. Orient edges from higher labels to lower; those edges running between identical labels are called *level*. Define a circuit in Γ_P (Γ_Q) to be a subgraph which is an imbedded circle. In Γ_p choose a point x in $S_P - \Gamma_P$ and define the interior of a circuit as the complementary region not containing x. In Γ_Q define the *interior* of a circuit as the complementary region not containing b_-. Further define interior vertex, chord, spoke, loop, base of loop, cycle, unicycle, semicycle, label sequence, interior label, sink, and source as in [Sc_1]. A *level circuit* is a circuit with all edges level. A level circuit is *pure* if every edge has the same label, otherwise a level circuit is *mixed.* The label of the edges of a pure



Labelling about a_i
Labelling about b_s similiar (change n to m)

FIGURE 2

level circuit is the *height* of that circuit. A *good* circuit, cycle, semicycle, or loop in Γ_Q is one whose interior does not contain b_+. A circuit, cycle, semicycle, or loop in Γ_Q whose interior does contain b_+ is *bad*.

1.6. Recall the following elementary facts, cf. [Sc_1, 2.5]:

Proposition. (1) *No chord of an innermost (good) cycle in* Γ_P (Γ_Q) *is oriented.*

(2) *Any chord of an innermost (good) semicycle in* Γ_P (Γ_Q) *is a level loop.*

(3) *If an innermost (good) cycle or semicycle in* Γ_P (Γ_Q) *has an interior vertex it must have an interior source or sink.*

(4) *Any (good) loop in* Γ_P (Γ_Q) *which has interior vertices either has an interior source or sink or else there is a cycle interior to the loop.*

2. Preliminary combinatorics

2.1 **Proposition.** *If* α *is a level edge in a (good) circuit in* Γ_P (Γ_Q) *then the labels which precede and follow the labels of* α *in the label sequence are equal to each other.*

Proof. See [Sc_1, 4.2].

2.2 **Proposition.** *A (good) unicycle in* Γ_P (Γ_Q) *has interior vertices.*

Proof. See [Sc_1, 4.5].

2.3 **Proposition.** *A good semicycle in* Γ_Q *has interior vertices.*

Proof. See [Sc_1, 4.7].

2.4 **Proposition.** *A cycle in* Γ_P *has interior vertices.*

Proof. See [Sc_2, 5.6].

2.5 **Proposition.** *A good loop in* Γ_Q *has interior vertices.*

Proof. Suppose there were an innermost good level loop α based at the vertex b_j in Γ_Q without interior vertices. The two ends of α either have label 1 or label m, without loss of generality, say m. Let D be the interior of α and denote $\partial D \cap \partial N$ by α'. The arc α' is an arc in b_j with endpoints the two labels m and meets no other label. Form a new planar surface P' by attaching the band B given by a regular neighborhood of α' in ∂N, see Figure 3.

The surface P' as boundary components $a_1, \cdots, a_{m-1}$ and two other components a_+, a_-, which (up to isotopy) are meridians of a 1-handle. By Figure 4, ∂D separates a_+ from a_- in P'.

Compressing P' via D gives two planar surfaces P_+, P_- each of which has a single boundary component on a 1-handle and any other boundary component parallel to D_h. Now it is easy to construct a simple closed curve on ∂N intersecting the boundary of P_+ in a single point, and hence intersecting in one point the sphere S in M obtained by attaching a disc to each component of ∂P_+ in N.

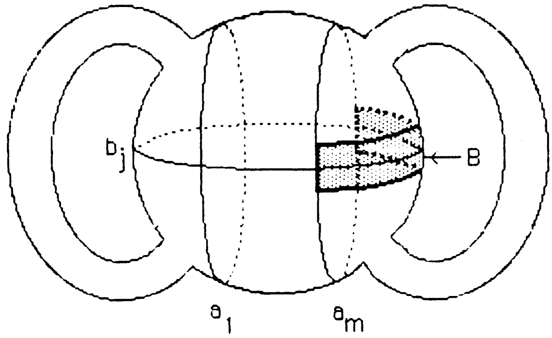


FIGURE 3

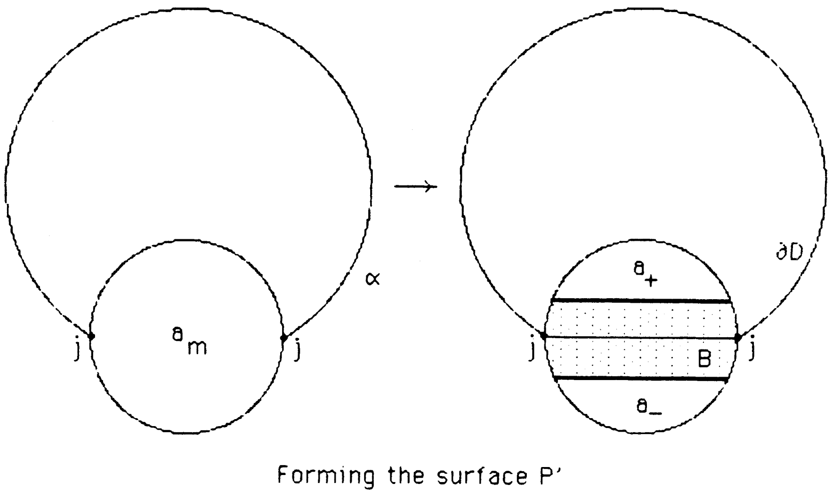


FIGURE 4

2.6 **Proposition.** *If* Γ_Q *contains a good loop then* Γ_P *contains a unicycle.*

Proof. See [Sc_1, 5.2].

2.7 **Proposition.** *If* Γ_Q *contains a good loop, then* Γ_Q *does not contain both a source and a sink.*

Proof. See [Sc_1, 5.4].

2.8 **Proposition.** *Good loops do not exist in* Γ_Q.

Proof. In the presence of 2.1–2.7, the path threading argument of [Sc_1, §6] and the minimality of $m + n$ show that there cannot be an innermost good semicycle in Γ_Q. So no good unicycle exists in Γ_Q. Similarly, no good level loop can exist in Γ_Q, as applying the path threading argument of [Sc_1, §6] to the interior of a good level loop in Γ_Q produces a good semicycle interior to that loop.

3. PURE LEVEL CIRCUITS IN Γ_P

3.1 **Lemma.** *A pure level circuit in Γ_P without oriented chords has interior vertices.*

3.2 *Proof of* 3.1. Suppose Ω is an innermost counterexample and that Ω has height k.

3.3 **Proposition.** Ω *has no chords.*

3.4 *Proof of* 3.3. Suppose α_0 is a level chord of Ω with label l. The ends of α_0 are on distinct vertices of Ω as a level loop would form a pure level circuit without oriented chords interior to Ω. By Figure 2, the other label l at either of these vertices is an interior label of Ω, and so must be incident to another level chord α_1, with label l. Continuing, one forms a pure level circuit without oriented chords interior to Ω.

3.5 *Continuation of* 3.2. As Ω has no chords, Ω is of height 1 or n. Without loss of generality, say n. Moreover, the interior of Ω is a disc, denote this disc by D.

Claim. ∂D lies entirely on one side of b_n.

Proof of claim. If $n > 1$, this is immediate. If $n = 1$, the claim follows from a slight modification of the argument in [Sc_1, 4.2].

Let D be the interior of Ω and let Q' be the planar surface obtained by attaching to Q the band B in ∂N which contains the "tops" of the vertices $a_1, \cdots, a_n$, see Figure 5.

The surface Q' has boundary $b_1, \cdots, b_{n-1}, b_+, b_-$, and two new boundary components b_* and $b_\#$ which are (up to isotopy) meridians of the 1-handles. Call any one of the circles b_+, b_-, b_*, or $b_\#$ a *1-handle boundary* of Q'.

Let $S_{Q'}$ be the two-sphere in S^3 obtained by capping off the planar surface Q' with discs in N. The surface $S_{Q'}$ is a Conway sphere for the composite knot K, decomposing S^3 into tangles T_1, in which D is a proper disc, and

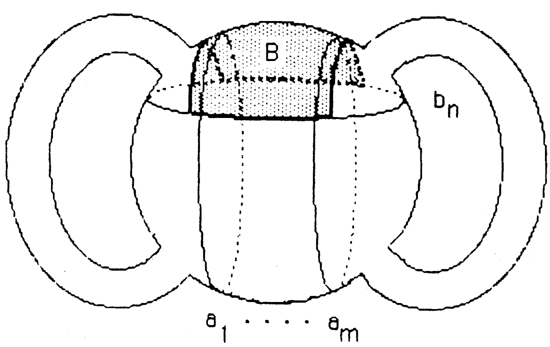


FIGURE 5

T_2 whose distinguished arcs are separated by the disc in N bounded by b_n. One of the distinguished arcs of T_2 is unknotted and the other distinquished arc is knotted according to one of the summands, say K_2, of K. It follows that the other summand K_1 of K is obtained by attaching an untangle, i.e. a trivial (3-ball, arc pair) to the tangle T_1.

3.6 **Proposition.** *The tangle* T_1 *is an untangle.*

3.7 *Proof of* 3.1 *from* 3.6. By assumption the bridge number of the summand K_1 is at least three. From 3.5 and 3.6 we conclude that the summand K_1 can be expressed as the union of two untangles. It follows that K_1 has bridge number at most two, a contradiction.

3.8 **Proposition.** *The distinquished arcs of* T_1 *are separated by the disc* D.

3.9 *Proof of* 3.6 *from* 3.8. It remains to show that there are no local knots in the distinquished arcs of T_1. Compressing the surface Q' via the disc D one obtains two planar surfaces Q^* and $Q^\#$. The planar surface Q^* $(Q^\#)$ in closure $(M - N)$ has exactly two boundary components which are meridians of the 1-handles (possibly of the same 1-handle) and at most $n-1$ other boundary components, each of which is parallel to ∂D_h. So the two-sphere S_{Q^*} $(S_{Q^\#})$ formed by capping of the surface Q^* $(Q^\#)$ with discs in N meets K in two points and separates S^3 into 3-balls B_1 and B_2. One of these balls, say B_1, is contained in the tangle T_1 and contains one of the two distinquished arcs (the other distinquished arc) of T_1. A local knot in either distinquished arc of T_1 would therefore contradict the minimality of $m + n$.

3.10 *Proof of* 3.8. Proposition 3.8 follows immediately from

3.11 **Proposition.** *No three of the* 1*-handle boundaries of* Q' *lie in the same component of* $Q' - \partial D$.

3.12 *Proof of* 3.11. Assume at least three of the four 1-handle boundaries of Q' lie in the same component of $Q' - \partial D$. There are two cases to consider.

Case 1. The circuit Ω is a level loop.

Proof of Case 1. By 2.8 the level loop α in Γ_Q corresponding to Ω is bad, and hence α separates b_+ from b_- in Γ_Q. As in the proof of 2.5 it follows that ∂D separates $\{b_+, b_*\}$ from $\{b_-, b_\#\}$, see Figure 6.

Case 2. The circuit Ω has more than one edge.

Proof of Case 2. It is essential to what follows to have a good picture of the family l of loops in Γ_Q corresponding to the edges of Ω in Γ_P and how their endpoints are distributed about b_n. A loop in l will be called an *l-loop* and a label on b_n incident to an l-loop an *l-label.* Both labels n on a vertex a_i of Ω in Γ_P are incident to edges of Ω, so both labels i on the vertex b_n in Γ_Q are l-labels. In particular, there is an equal number of l-labels on the front and back of b_n.

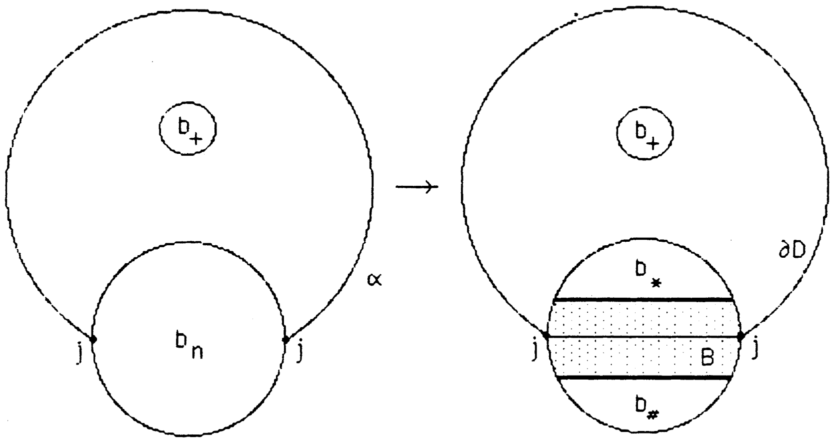


FIGURE 6

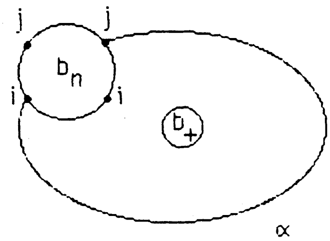


FIGURE 7

By 3.7, l is a family of bad loops based at a common vertex, so every l-loop separates b_+ from b_- in Γ_Q and the l-loops are linearly ordered by "interior".

3.13 **Definition.** A loop α in Γ_P is a *lobe* if both ends of α lie on either the front face or the back face of b_n. This is a slightly more general definition of a lobe than that of [BS, 6.1]

3.14 **Proposition.** *The inner- and outermost l-loops are lobes.*

3.15 *Proof of* 3.14. Consider a nonlobe α in l. As Ω is not a level loop, the labels i, j at the endpoints of α are not equal, say $i < j$. From Figure 7 we see that exactly one of the other l-labels i, j on b_n not incident to α is an interior label of α while the other is an exterior label.

Hence there are members of l interior and exterior to α.

3.16 *Continuation of* 3.12. Now refer to certain intervals on b_n as "points of the compass". In particular, call that interval meeting no label and having endpoints the two labels $n(1)$ north (south) and the interval incident to the inner (outer) most component of $Q - \{l \cup b_n\}$ as east (west). Call the four complementary intervals on b_n the NW, SW, SE, or NE *quadrant* of b_n as appropriate. Define NW, SW, SE, or NE (l-) *labels* in the obvious manner.

By 2.8, no l-loop joins NW to NE or SW to SE. Hence any nonlobe in l joins NE to SW or NW to SE. However, it is impossible to draw both types of nonlobes on the planar surface P, cf. [Sc_2, 3.2]. Without loss of generality then, assume all nonlobes in l join NE to SW, and hence that any member of l with a NW or SE label is a lobe. Proposition 3.14 implies that the number of l-labels in each quadrant is nonzero. The family l is as illustrated below in Figure 8.

To see how ∂D sits on Q', add the band B and recall that on B the curve ∂D is parallel to the core of B. Figure 9 and the convention above imply that the four 1-handle vertices of Q' correspond to the four cardinal directions N, S, E, and W on b_n. By the Jordan curve theorem, two 1-handle vertices are in the same component of $Q' - \partial D$ if and only if there is an arc joining the corresponding cardinal directions in b_n which contains an even number of l-labels.

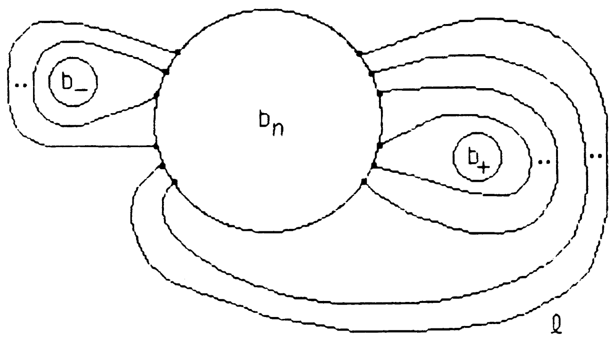


FIGURE 8

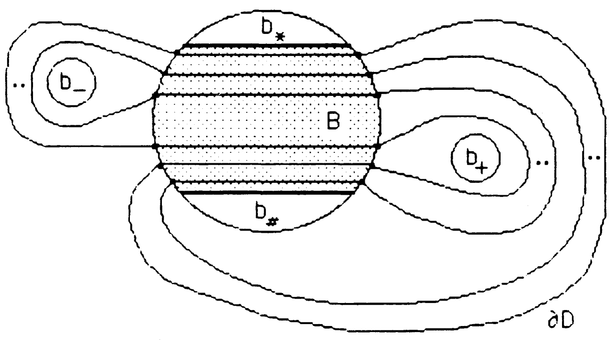


FIGURE 9

3.17 **Proposition.** *The curve ∂D separates b_+ from b_* in Q' if and only if ∂D separates b_- from $b_\#$ in Q'.*

3.18 **Corollary to 3.17.** *No 1-handle boundary in Q' is separated from the other three by ∂D.*

3.19 *Proof of* 3.17. There are an equal number of l-labels on the front and back face of b_n. As a nonlobe has one end on the front and one end on the back of b_n, the front and back face of b_n have an equal number of l-labels corresponding to nonlobes, hence equal numbers corresponding to lobes. From Figure 8, it follows that this number is exactly twice the number of NW (SE) l-labels. Hence there are an equal number of NW and SE l-labels. The curve ∂D separates b_+ from $b_\#$ (b_- from b_*) in Q' if and only if this number is odd.

3.20 *Continuation of* 3.12. By 3.18, all four of the 1-handle boundaries of Q' must lie in the same component of $Q' - \partial D$, so there must be an even number of l-labels in each quadrant. It follows that for some $k > 0$ there are $4k$ l-labels in all and so the circuit Ω has exactly $2k$ edges.

3.21 *The final contradiction.* We will contradict 3.20 by counting the edges of Ω another way. Consider traversing ∂D in Q' by beginning on and traveling in the direction of the arrow of the innermost loop in l with a NW label. Call an l-label where we pass from Q to B (B to Q) as we traverse ∂D an *exit* (*entry*) *point*, see Figure 10.

Map an exit (entry) point to the next exit (entry) point encountered as ∂D is traversed in the manner above to obtain a cyclic permutation of the $4k$ l-labels. As ∂D runs parallel to the core of b, an exit (entry) point moves a fixed number r of l-labels counter-clockwise (clockwise) about b_n under this permutation. From Figure 9, r equals twice the number of NW l-labels. As this latter number is even, $r = 4j$ for some $j > 0$.

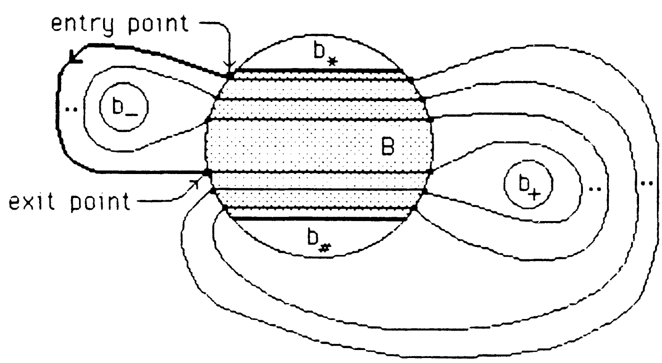


FIGURE 10

The number of edges in Ω is just the cardinality of an orbit of an exit (entry) point under this permutation. It follows that Ω has exactly

$$\frac{4k}{\text{g.c.d.}(4k, 4j)} \leq k \text{ edges}.$$

This contradicts 3.20.

4. x-CIRCUITS

4.1 **Lemma.** *There is either a cycle or a pure level circuit in Γ_P without chords and without interior vertices.*

Choose orientations for the planar surfaces P and Q, and let these induce orientations on their respective boundaries A_m and B_n. Call two components of A_m (B_n) parallel if their orientations are parallel in ∂N; if not call them antiparallel. There are two points of intersection of a_j with b_s, one on the front face and one on the back face of N. Label them respectively $f(i, s)$ and $r(i, s)$. As in [Sc$_1$, 2.2] we have the following "parity rule".

4.2 **Proposition.** *If an arc of $P \cap Q$ runs between $f(i, s)$ and $f(j, t)$, then one of the pairs a_j and a_j or b_s and b_t is parallel and the other is antiparallel. The same is true for an arc of $P \cap Q$ which runs between $r(i, s)$ and $r(j, t)$. On the other hand, if an arc of $P \cap Q$ runs between $f(i, s)$ and $r(j, t)$, then either both pairs are parallel or both are antiparallel.*

There are two classes of vertices in Γ_P, each consisting of parallel vertices. A typical vertex in each class is illustrated in Figure 11, where $f(s)$ $(r(t))$ denotes that the label s (t) lies on the front (back) face of N.

We form a new graph Γ_{P^*} from the graph Γ_P as follows: Reenumerate the labels of the vertices in the first class given above by changing $f(i)$ to i, and $r(j)$ to $2n + 1 - j$, see Figure 12.

Similarly, reenumerate the labels of the vertices of the second class by changing $r(i)$ to i, and $f(j)$ to $2n + 1 - j$, see Figure 13.

By construction, therefore, all the vertices in Γ_{P^*} are parallel. As before, orient edges from higher labels to lower; and continue to call level those edges running between identical labels. Proposition 4.2 now implies the following.

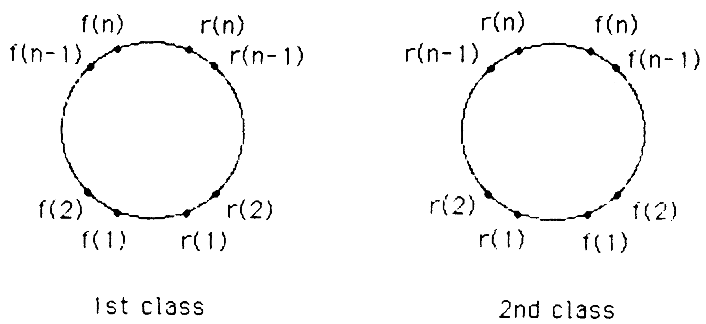


FIGURE 11

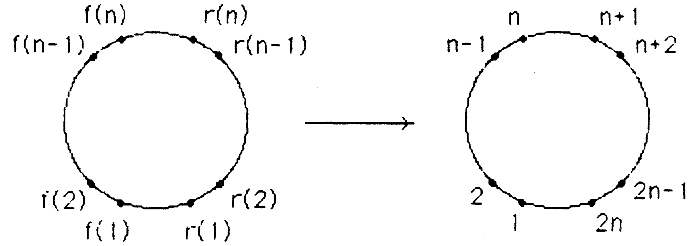


FIGURE 12

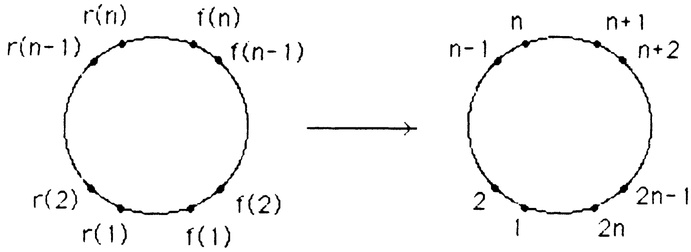


FIGURE 13

4.3 **Proposition.** *There are no level edges in* Γ_{P^*}.

For $x \in \{1, 2, \dots, 2n\}$ define an x-circuit to be a circuit which can be traversed in a direction so that the initial point of every edge has label x.

4.4 **Proposition.** *In* Γ_{P^*} *there is an* x*-circuit without interior vertices and without chords.*

4.5 *Proof of* 4.4. (cf. [CLGS, 2.6.2]). Choose an $x \in \{1, 2, \dots, 2n\}$ and construct a path beginning at some vertex v, always choosing the label x as the initial point of each edge; by 4.3 this path can be followed until a vertex is repeated, forming an x-circuit. Choose an innermost such circuit (varying over all x), and denote it by Ω and suppose that Ω has interior vertices or chords.

If Ω has no interior labels but has interior vertices, then for any $y \in [1, 2, \dots, 2n\}$ it is possible to find a y-circuit interior to Ω. So suppose Ω has interior labels. Choose a vertex v in Ω, and choose the label on v which is adjacent to x and is either an interior label of Ω or a label of an edge in the circuit. This label is either $x-1$ or $x+1 \pmod{2n}$, without loss of generality, suppose it is $x-1$. Since all the vertices of Γ_{P^*} are parallel, for each vertex of Ω the label $x-1$ is either an interior label or a label of an edge in the circuit. The circuit Ω has interior labels, so there is at least one vertex u in Ω for which the label $x-1$ is an interior label. Beginning at u, we can construct a path in which every edge has initial label $x-1$; when a vertex is repeated we obtain an $(x-1)$-circuit, call it Ω'.

The circuit Ω' must be interior to Ω as each vertex in Ω or in its interior either has the label $x-1$ as an end of an edge of Ω, or as an end of an interior edge. The circuit $\Omega' \neq \Omega$ as the edge of Ω incident to u with label at u different from x cannot be part of Ω' as its label at u is also different from the interior label $x-1$. This contradicts the fact that Ω was innermost, and 4.4 follows.

4.6 *Proof of* 4.1. The x-circuit in Γ_{P^*} given by 4.4 has a consistent orientation of its edges so that the initial point of every edge has label x and the terminal point of every edge has label $x-1$. It follows that in Γ_P this circuit is either a cycle or a pure level circuit.

4.7 *Proof of* 1.3. A contradiction exists between 2.4, 3.1, and 4.1.

5. Concluding remarks

5.1. A ribbon disc for K^2, the connected sum of a knot K with its mirror image, is obtained by spinning an appropriately knotted arc in R^3_+. If K has bridge number b, this ribbon disc has $b-1$ saddle points. Theorem 1.3 shows that the ribbon number of K^2 is $b-1$ for $b<4$. So it seems appropriate to conjecture that the ribbon number of K^2 always equals the bridge number of K minus 1.

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